ISRAEL JOURNAL OF MATHEMATICS **156** (2006), 359–380

MULTIPLE SOLUTIONS TO SINGULAR CRITICAL ELLIPTIC EQUATIONS

BY

PIGONG HAN

Institute of Applied Mathematics, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100080, P. R. of China
e-mail: pghan@amss.ac.cn

ABSTRACT

Assume $0 \leq \mu < \bar{\mu} = (\frac{N-2}{2})^2$ and let $\Omega \subset R^N (N \geq 4)$ be a smooth bounded domain, $0 \in \Omega$. We study the semilinear elliptic problem: $-\Delta u - \mu\frac{u}{|x|^2} = \lambda u + Q(x)|u|^{2^*-2}u, u \in H_0^1(\Omega)$. By investigating the effect of the coefficient Q, we establish the existence of nontrivial solutions for any $\lambda > 0$ and multiple positive solutions with $\lambda, \mu > 0$ small.

1. Introduction and main results

Let $\Omega \subset R^N (N \geq 4)$ be an open bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $2^* = \frac{2N}{N-2}$. We are concerned with the following semilinear elliptic problem,

$$(1.1) \qquad \begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = \lambda u + Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $Q(x)$ is a positive bounded function on $\overline{\Omega}$, $\lambda > 0$ and $0 \leq \mu < \bar{\mu} = (\frac{N-2}{2})^2$.

$u \in H_0^1(\Omega)$ is said to be a weak solution of problem (1.1) if u satisfies

$$(1.2) \qquad \int_\Omega (\nabla u \cdot \nabla v - \mu\frac{uv}{|x|^2} - \lambda uv - Q(x)|u|^{2^*-2}uv)dx = 0 \quad \forall v \in H_0^1(\Omega).$$

It is well known that the nontrivial solutions of problem (1.1) are equivalent to the nonzero critical points of the energy functional

$$(1.3) \quad I_{\lambda,\mu}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \mu\frac{u^2}{|x|^2} - \lambda u^2)dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}dx, \quad u \in H_0^1(\Omega).$$

Received March 8, 2005

In recent years, much attention has been paid to the existence of nontrivial solutions of problem (1.1) (see [2, 3, 4, 8, 10, 11, 14]). Let σ_μ denote the spectrum of the operator $-\Delta - \frac{\mu}{|x|^2}(0 \le \mu < \bar{\mu})$ with zero Dirichlet boundary condition. In view of [6, 9], $\sigma_\mu(0 \le \mu < \bar{\mu})$ is discrete, contained in the positive semi-axis and each eigenvalue $\lambda_{\mu,i}(i = 1, 2, \dots)$ is isolated and has finite multiplicity, the smallest eigenvalue $\lambda_{\mu,1}$ being simple and $\lambda_{\mu,i} \longrightarrow \infty$ as $i \longrightarrow \infty$; moreover, each L^2-normalized eigenfunction $e_{\mu,i}$ corresponding to $\lambda_{\mu,i} \in \sigma_\mu$, belongs to the space $H_0^1(\Omega)$.

The functional $I \in C^1(X, R)$ is said to satisfy the $(P.S.)_c$ condition if any sequence $\{u_n\} \subset X$ such that as $n \longrightarrow \infty$

$$I(u_n) \to c, \quad dI(u_n) \to 0 \quad \text{strongly in } X^*$$

contains a subsequence converging in X to a critical point of I. In this paper, we will take $I = I_{\lambda,\mu}$ and $X = H_0^1(\Omega)$.

Set $D^{1,2}(R^N) = \{u \in L^{2^*}(R^N) | \ |\nabla u| \in L^2(R^N)\}$. For all $\mu \in [0, \bar{\mu})$, $\bar{\mu} = (\frac{N-2}{2})^2$, we define the constant

$$S_\mu := \inf_{u \in D^{1,2}(R^N)\backslash\{0\}} \frac{\int_{R^N}(|\nabla u|^2 - \mu\frac{u^2}{|x|^2})dx}{(\int_{R^N}|u|^{2^*}dx)^{\frac{2}{2^*}}}.$$

From [9, 11], S_μ is independent of any $\Omega \subset R^N$ in the sense that if

$$S_\mu(\Omega) := \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega(|\nabla u|^2 - \mu\frac{u^2}{|x|^2})dx}{(\int_\Omega|u|^{2^*}dx)^{\frac{2}{2^*}}},$$

then $S_\mu(\Omega) = S_\mu(R^N) = S_\mu$.

Let $\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}, \gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$, S. Terracini [15] proved that for $\epsilon > 0$,

$$U_{\mu,\epsilon}(x) = \frac{(4\epsilon^2 N(\bar{\mu} - \mu)/(N-2))^{\frac{N-2}{4}}}{(\epsilon^2|x|^{\frac{\gamma'}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\gamma}{\sqrt{\bar{\mu}}}})^{\sqrt{\bar{\mu}}}} \tag{1.4}$$

satisfies

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = |u|^{2^*-2}u & \text{in } R^N\backslash\{0\}, \\ u \longrightarrow 0 & \text{as } |x| \longrightarrow \infty. \end{cases} \tag{1.5}$$

From Theorem B in [5], all the positive solutions of problem (1.5) must have the form of $U_{\mu,\epsilon}$. Moreover, $U_{\mu,\epsilon}$ achieves S_μ.

By the Hardy inequality (see [1])

$$\int_\Omega \frac{u^2}{|x|^2}dx \le \frac{1}{\bar\mu}\int_\Omega |\nabla u|^2 dx \quad \forall u \in H_0^1(\Omega),$$

we easily derive that the norm $(\int_\Omega(|\nabla u|^2 - \mu\frac{u^2}{|x|^2})dx)^{\frac{1}{2}}$ $(0<\mu<\bar\mu)$ is equivalent to the usual norm in $H_0^1(\Omega)$.

In a recent paper, D. Cao and P. Han [3] considered a special case of problem (1.1) (i.e. $Q(x)\equiv const$; without loss of generality, assume $Q(x)\equiv 1$). Namely, for

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

they proved that: *Assume that* $0\le\mu<(\frac{N-2}{2})^2-(\frac{N+2}{N})^2$, *then for all* $\lambda>0$, *problem (1.6) admits a nontrivial solution with critical level in the range* $(0,\frac{1}{N}S_\mu^{\frac{N}{2}})$.

When $Q(x)\not\equiv const$, the analysis of Palais–Smale sequences becomes complicated, which results in much difficulty. It is natural to ask whether problem (1.1) admits one solution for any $\lambda>0$. In the present note, we not only give a positive answer, but also prove the multiplicity of positive solutions for $\lambda,\mu>0$ small.

In this paper, we suppose that $Q(x)$ is a positive bounded function on $\overline\Omega$. Moreover,

(H_1) $Q(x)=Q(0)+O(|x|^2)$ as $x\to 0$.

(H_2) There exist points $a_1,a_2,\dots,a_k\in\Omega\backslash\{0\}$ such that $Q(a_i)$ are strict local maxima satisfying

$$Q(a_i)=Q_M=\max_{\overline\Omega}Q(x)>0,$$

and

$$Q(x)=Q(a_i)+o(|x-a_i|^2)\quad \text{as } x\to a_i, 1\le i\le k.$$

In order to state our main results, we need to distinguish two cases:

Case I: $Q(0)\ge Q_M(\frac{S_\mu}{S_0})^{\frac{N}{N-2}}$;

Case II: $Q(0)< Q_M(\frac{S_\mu}{S_0})^{\frac{N}{N-2}}$.

Theorem 1.1: *In* Case I. *Assume that* $0\le\mu<\bar\mu-(\frac{N+2}{N})^2 (N\ge 5)$ *and* (H_1) *holds. Then, for all* $\lambda>0$ *problem (1.1) admits a nontrivial solution* u *such that* $I_{\lambda,\mu}(u)\in(0,S_\mu^{\frac{N}{2}}/NQ(0)^{\frac{N-2}{2}})$.

THEOREM 1.2: *In Case II. Let $N \geq 5$, $0 \leq \mu < \bar{\mu}$ and (H_2) hold. Then, for all $\lambda > 0$ problem (1.1) has at least one solution v such that $I_{\lambda,\mu}(v) \in (0, S_0^{\frac{N}{2}}/NQ_M^{\frac{N-2}{2}})$.*

Furthermore, by analyzing the effect of the coefficient $Q(x)$, we obtain the multiplicity of positive solutions of (1.1) for $\lambda, \mu > 0$ small.

THEOREM 1.3: *In Case II. Suppose $N \geq 4$ and $(H_1) - (H_2)$ hold. Then there exist $\mu_0 > 0, \lambda_0 > 0$ such that for $\mu \in (0, \mu_0)$, problem (1.1) admits at least k positive solutions with all $\lambda \in (0, \lambda_0)$.*

We prove Theorems 1.1, 1.2 and 1.3 by critical point theory. However, the functional $I_{\lambda,\mu}$ does not satisfy the Palais–Smale (*(P.S.)* in short) condition due to the lack of compactness of the embeddings: $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega, |x|^{-2})$. So the standard variational argument is not applicable directly; we need to analyze the effect of the coefficient Q and the energy range where $I_{\lambda,\mu}$ satisfies the Palais–Smale condition. We prove the existence of nontrivial solutions for any $\lambda > 0$ and multiple positive solutions of problem (1.1) with $\lambda > 0, \mu > 0$ small by the linking theorem and mountain pass lemma (see [13, 16]).

Throughout this paper, we denote the norm of $H_0^1(\Omega)$ by $\| u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$, the norm of $L^l(\Omega)(1 \leq l < \infty)$ by $\| u\|_{L^l(\Omega)} = (\int_\Omega |u|^l dx)^{\frac{1}{l}}$ and positive constants (possibly different) by $C, C_1, C_2, \ldots$.

2. Proof of Theorem 1.1

In this section, we first introduce some preliminary lemmas.

LEMMA 2.1: *Let $0 \leq \mu < \bar{\mu}$. Then for every $\lambda > 0$, $I_{\lambda,\mu}$ satisfies the $(P.S.)_c$ condition with $c < c^*$, where*

$$c^* = \min\left\{ \frac{S_\mu^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}, \frac{S_0^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}} \right\}.$$

Proof: Assume that $\{u_n\} \subset H_0^1(\Omega)$ satisfies, as $n \longrightarrow \infty$,

$$I_{\lambda,\mu}(u_n) \longrightarrow c < c^*, dI_{\lambda,\mu}(u_n) \longrightarrow 0 \text{ strongly in } H^{-1}(\Omega).$$

By the Hardy inequality, we easily get $\| u_n\| \leq C$. Therefore, up to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega);\\ &u_n \rightharpoonup u \text{ weakly in } L^2(\Omega, |x|^{-2}dx);\\ &u_n \rightharpoonup u \text{ weakly in } L^{2^*}(\Omega);\\ &u_n \longrightarrow u \text{ strongly in } L^2(\Omega);\\ &u_n \longrightarrow u \text{ a.e. on } \Omega. \end{aligned}$$

It is easy to verify that $u \in H_0^1(\Omega)$ is a weak solution of problem (1.1).

Hence, by the concentration compactness principle [12], there exists a subsequence, still denoted by $\{u_n\}$, at most countable set $\mathcal{J}$, a set of different points $\{x_j\}_{j\in\mathcal{J}}$, and $\{\widetilde{\mu_j}\}_{j\in\mathcal{J}\cup\{0\}}$, $\{\widetilde{\nu_j}\}_{j\in\mathcal{J}\cup\{0\}} \subset [0,\infty)$ such that

$$\begin{aligned} &|\nabla u_n|^2 \rightharpoonup d\widetilde{\mu} \geq |\nabla u|^2 + \sum_{j\in\mathcal{J}} \widetilde{\mu_j}\delta_{x_j} + \widetilde{\mu_0}\delta_0,\\ &|u_n|^{2^*} \rightharpoonup d\widetilde{\nu} = |u|^{2^*} + \sum_{j\in\mathcal{J}} \widetilde{\nu_j}\delta_{x_j} + \widetilde{\nu_0}\delta_0,\\ &\frac{|u_n|^2}{|x|^2} \rightharpoonup d\widetilde{\gamma} = \frac{|u|^2}{|x|^2} + \widetilde{\gamma_0}\delta_0,\\ &S_0\widetilde{\nu_j}^{\frac{2}{2^*}} \leq \widetilde{\mu_j} \quad \text{for } j \in \mathcal{J},\\ &S_\mu\widetilde{\nu_0}^{\frac{2}{2^*}} \leq \widetilde{\mu_0} - \mu\widetilde{\gamma_0}. \end{aligned}$$

We claim that $\mathcal{J}$ is finite and that for any $j \in \mathcal{J}$, either $\widetilde{\nu_j} = 0$ or

$$Q(x_j)\widetilde{\nu_j} \geq S_0^{\frac{N}{2}}/Q_M^{\frac{N-2}{2}}.$$

In fact, let $\epsilon > 0$ be small enough such that $0 \notin B_\epsilon(x_j)(j \in \mathcal{J})$. Let ϕ^j be a smooth cut off function centered at x_j satisfying

$$0 \leq \phi^j \leq 1, \phi^j(x) = \begin{cases} 1 & \text{if } |x - x_j| \leq \frac{\epsilon}{2},\\ 0 & \text{if } |x - x_j| \geq \epsilon, \end{cases} \quad \text{and } |\nabla\phi^j| \leq \frac{4}{\epsilon}.$$

Observe that

$$\begin{aligned} \langle dI_{\lambda,\mu}(u_n), u_n\phi^j\rangle = &\int_\Omega |\nabla u_n|^2\phi^j dx + \int_\Omega u_n\nabla u_n\nabla\phi^j dx - \mu\int_\Omega \frac{|u_n|^2\phi^j}{|x|^2}dx\\ &- \lambda\int_\Omega |u_n|^2\phi^j dx - \int_\Omega Q(x)|u_n|^{2^*}\phi^j dx, \end{aligned} \tag{2.1}$$

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2\phi^j dx = \int_\Omega \phi^j d\widetilde{\mu} \geq \int_\Omega |\nabla u|^2\phi^j dx + \widetilde{\mu_j}, \tag{2.2}$$

$$\lim_{n\to\infty}\int_\Omega Q(x)|u_n|^{2^*}\phi^j dx=\int_\Omega Q(x)\phi^j d\widetilde{\nu}=\int_\Omega Q(x)|u|^{2^*}\phi^j dx+Q(x_j)\widetilde{\nu_j}, \tag{2.3}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega u_n\nabla u_n\nabla\phi^j dx\right| \\
&\le \lim_{\epsilon\to 0}\lim_{n\to\infty}\left(\left(\int_\Omega|\nabla u_n|^2dx\right)^{\frac12}\left(\int_\Omega|u_n|^2|\nabla\phi^j|^2dx\right)^{\frac12}\right)\\
&\le C\lim_{\epsilon\to 0}\left(\int_\Omega|u|^2|\nabla\phi^j|^2dx\right)^{\frac12}\\
&\le C\lim_{\epsilon\to 0}\left(\left(\int_{B_\epsilon(x_j)}|\nabla\phi^j|^N dx\right)^{\frac1N}\left(\int_{B_\epsilon(x_j)}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\right)\\
&\le C\lim_{\epsilon\to 0}\left(\int_{B_\epsilon(x_j)}|u|^{2^*}dx\right)^{\frac{1}{2^*}}\\
&=0,
\end{aligned} \tag{2.4}$$

and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega\frac{|u_n|^2\phi^j}{|x|^2}dx=0,\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega|u_n|^2\phi^j dx=0. \tag{2.5}$$

Inserting (2.2)–(2.5) into (2.1), we deduce

$$0=\lim_{\epsilon\to 0}\lim_{n\to\infty}\langle dI_{\lambda,\mu}(u_n),u_n\phi^j\rangle\ge\widetilde{\mu_j}-Q(x_j)\widetilde{\nu_j}. \tag{2.6}$$

Since $S_0\widetilde{\nu_j}^{\frac{2}{2^*}}\le\widetilde{\mu_j}$ for $j\in\mathcal{J}$, together with (2.6), we infer that $\widetilde{\nu_j}=0$ or $Q(x_j)\widetilde{\nu_j}\ge S_0^{\frac N2}/Q_M^{\frac{N-2}{2}}$, which implies that $\mathcal{J}$ is finite.

Now we consider the possibility of concentration at the origin. Let $\epsilon>0$ be small enough such that $x_j\notin B_\epsilon(0)(j\in\mathcal{J})$. Let ϕ be a smooth cut off function centered at 0 satisfying

$$0\le\phi\le 1,\phi(x)=\begin{cases}1 & \text{if } |x|\le\frac{\epsilon}{2},\\ 0 & \text{if } |x|\ge\epsilon,\end{cases}\quad\text{and } |\nabla\phi|\le\frac{4}{\epsilon}.$$

Then we have

$$\lim_{n\to\infty}\int_\Omega|\nabla u_n|^2\phi dx=\int_\Omega\phi d\widetilde{\mu}\ge\int_\Omega|\nabla u|^2\phi dx+\widetilde{\mu_0},$$

$$\lim_{n\to\infty}\int_\Omega Q(x)|u_n|^{2^*}\phi dx=\int_\Omega Q(x)\phi d\widetilde{\nu}=\int_\Omega Q(x)|u|^{2^*}\phi dx+Q(0)\widetilde{\nu_0},$$

$$\lim_{n\to\infty}\int_\Omega\frac{|u_n|^2\phi}{|x|^2}dx=\lim_{n\to\infty}\int_\Omega\phi d\widetilde{\gamma}=\int_\Omega\frac{|u|^2\phi}{|x|^2}dx+\widetilde{\gamma_0},$$

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n\nabla u_n\nabla\phi dx=0,$$

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^2\phi dx=0.$$

Hence, we conclude that

$$0=\lim_{\epsilon\to 0}\lim_{n\to\infty}\langle dI_{\lambda,\mu}(u_n),u_n\phi\rangle\geq\widetilde{\mu_0}-\mu\widetilde{\gamma_0}-Q(0)\widetilde{\nu_0}. \tag{2.7}$$

Since $S_\mu\widetilde{\nu_0}^{\frac{2}{2^*}}\leq\widetilde{\mu_0}-\mu\widetilde{\gamma_0}$, together with (2.7), we get

$$S_\mu\widetilde{\nu_0}^{\frac{2}{2^*}}\leq Q(0)\widetilde{\nu_0},$$

which implies that $\widetilde{\nu_0}=0$ or $\widetilde{\nu_0}\geq(\frac{S_\mu}{Q(0)})^{\frac{N}{2}}$.

From the above arguments, we conclude

$$\begin{aligned}c&=I_{\lambda,\mu}(u_n)-\frac{1}{2}\langle dI_{\lambda,\mu}(u_n),u_n\rangle+o(1)\\&=\frac{1}{N}\int_\Omega Q(x)|u_n|^{2^*}dx+o(1)\\&=\frac{1}{N}\Big(\int_\Omega Q(x)|u|^{2^*}dx+\sum_{j\in\mathcal{J}}Q(x_j)\widetilde{\nu_j}+Q(0)\widetilde{\nu_0}\Big).\end{aligned}$$

If there is a $j\in\mathcal{J}\cup\{0\}$ such that $\widetilde{\nu_j}\neq 0$, then we infer that

$$c\geq\min\left\{\frac{S_\mu^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}},\frac{S_0^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}\right\}=c^*,$$

which contradicts the assumption on c.

Hence, up to a subsequence, we derive that $u_n\longrightarrow u$ strongly in $H_0^1(\Omega)$. ∎

Denote by $B_r(y)$ the ball of radius r centered at the point $y\in\Omega$; we have $B_{\frac{2}{m}}(y)\subset\Omega$ for m large enough. For $0\leq\mu<\bar{\mu}$, let

$$H^-=span\{e_{\mu,1},e_{\mu,2},\ldots,e_{\mu,k}\},\quad H^+=(H^-)^\perp.$$

Fix k, define the approximating eigenfunctions $e_{\mu,i}^m=\xi_m e_{\mu,i}(i=1,2,\ldots)$ and the space

$$H_m^-=span\{e_{\mu,1}^m,e_{\mu,2}^m,\ldots,e_{\mu,k}^m\},$$

where

$$\xi_m(x)=\begin{cases}0 & \text{if } x\in B_{\frac{1}{m}}(0),\\ m|x|-1 & \text{if } x\in B_{\frac{2}{m}}(0)\backslash B_{\frac{1}{m}}(0),\\ 1 & \text{if } x\in\Omega\backslash B_{\frac{2}{m}}(0).\end{cases}$$

We have the following error estimates, which can be found in [3]:

LEMMA 2.2: *Let $0 \leq \mu < \bar{\mu}$. Then*

(i) $| e^m_{\mu,i} - e_{\mu,i}| \longrightarrow 0$ as $m \longrightarrow \infty$;

(ii) $\max_{\{u \in H^-_m, |u|_{L^2(\Omega)}=1\}} |u|^2 \leq \lambda_{\mu,k} + Cm^{-2\sqrt{\bar{\mu}-\mu}}$.

For any $m > 0, \epsilon > 0$, we define

$$(2.8) \qquad u^m_\epsilon(x) = \begin{cases} U_{\mu,\epsilon}(x) - \frac{(4\epsilon^2 N(\bar{\mu}-\mu)/(N-2))^{\frac{N-2}{4}}}{(\epsilon^2 (\frac{1}{m})^{\frac{\gamma'}{\sqrt{\bar{\mu}}}} + (\frac{1}{m})^{\frac{\gamma}{\sqrt{\bar{\mu}}}})^{\sqrt{\bar{\mu}}}} & \text{if } x \in B_{\frac{1}{m}}(0), \\ 0 & \text{if } x \in \Omega \backslash B_{\frac{1}{m}}(0). \end{cases}$$

The following estimates hold (see [9]): For any $0 \leq \mu < \bar{\mu}$,

$$(2.9) \qquad \int_\Omega \Big(|\nabla u^m_\epsilon|^2 - \mu \frac{(u^m_\epsilon)^2}{|x|^2} \Big) dx \leq S^{\frac{N}{2}}_\mu + C_1 \epsilon^{N-2} m^{2\sqrt{\bar{\mu}-\mu}},$$

$$(2.10) \qquad \int_\Omega |u^m_\epsilon|^{2^*} dx \geq S^{\frac{N}{2}}_\mu - C_2 \epsilon^N m^{\frac{2N}{N-2}\sqrt{\bar{\mu}-\mu}}.$$

Set

$$c_\epsilon = \inf_{h \in \Gamma_{\epsilon,m}} \max_{u \in Q_{\epsilon,m}} I_{\lambda,\mu}(h(u)),$$

where

$$\Gamma_{\epsilon,m} = \{h \in C(Q_{\epsilon,m}, H^1_0(\Omega)) | \ h(u) = u, \forall u \in \partial Q_{\epsilon,m}\}$$

and

$$Q_{\epsilon,m} = (\overline{B_R(0)} \cap H^-_m) \oplus \{r u^m_\epsilon | \ 0 \leq r \leq R\}.$$

Then we have the following:

LEMMA 2.3: *Let the assumption (H_1) hold and $\mu \in [0, \bar{\mu} - (\frac{N+2}{N})^2)$. Then for any $\lambda > 0$, $c_\epsilon < S^{\frac{N}{2}}_\mu / NQ(0)^{\frac{N-2}{2}}$.*

Proof: Without loss of generality, we may assume that there exists an integer k such that $\lambda_{\mu,k} \leq \lambda < \lambda_{\mu,k+1}$. Let $\max_{u \in Q_{\epsilon,m}} I_{\lambda,\mu}(u) = I_{\lambda,\mu}(w^m_\mu + t^m_{\mu,\epsilon} u^m_\epsilon)$, where $w_\mu^m \in H_m^-$. By (ii) of Lemma 2.2, we get

(2.11)

$$\begin{aligned}
I_{\lambda,\mu}(w_\mu^m) =& \frac{1}{2}\int_\Omega \Big(|\nabla w_\mu^m|^2 - \mu\frac{(w_\mu^m)^2}{|x|^2} - \lambda (w_\mu^m)^2\Big)dx - \frac{1}{2^*}\int_\Omega Q(x)|w_\mu^m|^{2^*}dx \\
\leq& \frac{\lambda_{\mu,k}-\lambda}{2}\int_\Omega (w_\mu^m)^2 dx + Cm^{-2\sqrt{\bar\mu-\mu}}\int_\Omega (w_\mu^m)^2 dx - \frac{1}{2^*}\min_{\overline{\Omega}} Q(x) \\
& \int_\Omega |w_\mu^m|^{2^*}dx \\
\leq& Cm^{-2\sqrt{\bar\mu-\mu}}|\, w_\mu^m \,|^2_{L^{2^*}(\Omega)} - \frac{1}{2^*}\min_{\overline{\Omega}} Q(x)|\, w_\mu^m \,|^{2^*}_{L^{2^*}(\Omega)} \\
\leq& \max_{t\geq 0}(Cm^{-2\sqrt{\bar\mu-\mu}}t^2 - \frac{1}{2^*}\min_{\overline{\Omega}} Q(x) t^{2^*}) \\
\leq& Cm^{-N\sqrt{\bar\mu-\mu}}.
\end{aligned}$$

On the other hand, as in [9], choose $\epsilon = m^{-\frac{N+2}{N-2}\sqrt{\bar\mu-\mu}}$. Thus as $m \longrightarrow \infty$, (2.9) and (2.10) become respectively

$$\int_\Omega \Big(|\nabla u_\epsilon^m|^2 - \mu\frac{(u_\epsilon^m)^2}{|x|^2}\Big)dx \leq S_\mu^{\frac{N}{2}} + C_1 m^{-N\sqrt{\bar\mu-\mu}}, \tag{2.12}$$

$$\int_\Omega |u_\epsilon^m|^{2^*}dx \geq S_\mu^{\frac{N}{2}} - C_2 m^{-\frac{N^2}{N-2}\sqrt{\bar\mu-\mu}}. \tag{2.13}$$

From (2.13) and the assumption of (H_1), we easily deduce that for m large enough

$$\int_\Omega Q(x)|u_\epsilon^m|^{2^*}dx \geq Q(0)S_\mu^{\frac{N}{2}} - C_3 m^{-\frac{N^2}{N-2}\sqrt{\bar\mu-\mu}}. \tag{2.14}$$

Furthermore,

$$\int_\Omega |u_\epsilon^m|^2 dx \geq C_4 m^{-(N+2)}. \tag{2.15}$$

Observe that $id \in \Gamma_{\epsilon,m}$ and $|supp w_\mu^m \cap supp u_\epsilon^m| = 0$. From (2.11), (2.12), (2.14) and (2.15), we conclude that

$$
\begin{aligned}
c_\epsilon \leq & \max_{u\in Q_{\epsilon,m}} I_{\lambda,\mu}(u)\\
=& I_{\lambda,\mu}(w_\mu^m + t_{\mu,\epsilon}^m u_\epsilon^m)\\
=& I_{\lambda,\mu}(w_\mu^m) + I_{\lambda,\mu}(t_{\mu,\epsilon}^m u_\epsilon^m)\\
\leq & Cm^{-N\sqrt{\bar\mu-\mu}} + \frac{(t_{\mu,\epsilon}^m)^2}{2}\int_\Omega \left(|\nabla u_\epsilon^m|^2 - \mu\frac{(u_\epsilon^m)^2}{|x|^2} - \lambda(u_\epsilon^m)^2\right)dx\\
&- \frac{(t_{\mu,\epsilon}^m)^{2^*}}{2^*}\int_\Omega Q(x)|u_\epsilon^m|^{2^*}dx\\
\leq & Cm^{-N\sqrt{\bar\mu-\mu}} + \frac{(t_{\mu,\epsilon}^m)^2}{2}(S_\mu^{\frac{N}{2}} + C_1 m^{-N\sqrt{\bar\mu-\mu}} - \lambda C_4 m^{-(N+2)})\\
&- \frac{(t_{\mu,\epsilon}^m)^{2^*}}{2^*}\left(Q(0)S_\mu^{\frac{N}{2}} - C_3 m^{-\frac{N^2}{N-2}\sqrt{\bar\mu-\mu}}\right)\\
\leq & Cm^{-N\sqrt{\bar\mu-\mu}} + \frac{1}{N}(S_\mu^{\frac{N}{2}} + C_1 m^{-N\sqrt{\bar\mu-\mu}} - \lambda C_4 m^{-(N+2)})\\
&\times\left(\frac{S_\mu^{\frac{N}{2}} + C_1 m^{-N\sqrt{\bar\mu-\mu}} - \lambda C_4 m^{-(N+2)}}{Q(0)S_\mu^{\frac{N}{2}} - C_3 m^{-\frac{N^2}{N-2}\sqrt{\bar\mu-\mu}}}\right)^{\frac{N-2}{2}},
\end{aligned} \tag{2.16}
$$

where we use the following fact:

$$\max_{t\geq 0}\left(\frac{t^2}{2}A - \frac{t^{2^*}}{2^*}B\right) = \frac{1}{N}A\left(\frac{A}{B}\right)^{\frac{N-2}{2}}, \quad A,B>0.$$

Note that $0\leq\mu<\bar\mu-(\frac{N+2}{N})^2$, and then $N+2<N\sqrt{\bar\mu-\mu}<\frac{N^2}{N-2}\sqrt{\bar\mu-\mu}$. Hence, for m large enough, we deduce from (2.16) that

$$c_\epsilon \leq \frac{S_\mu^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}} + Cm^{-N\sqrt{\bar\mu-\mu}} - C_5 m^{-(N+2)} < \frac{S_\mu^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}. \qquad ∎$$

Proof of Theorem 1.1: From [9], for m,R large enough $I_{\lambda,\mu}$ satisfies all the assumptions of the linking theorem [13] except for the $(P.S.)_c$ condition, i.e.,

(i) There exist $\alpha_0,\rho_0>0$ such that

$$I_{\lambda,\mu}(u)\geq\alpha_0 \quad \forall u\in\partial B_{\rho_0}(0)\cap H^+.$$

(ii) There exists $R_0>\rho_0$ such that

$$I_{\lambda,\mu}|_{\partial Q_{\epsilon,m}}\leq\omega(m) \quad \text{with } \omega(m)\longrightarrow 0 \text{ as } m\longrightarrow\infty.$$

Moreover, $\partial B_{\rho_0}(0)\cap H^+$ and $\partial Q_{\epsilon,m}$ link (cf. [13]). Then we obtain a Palais–Smale sequence $\{u_n\}$ for $I_{\lambda,\mu}$ at level c_ϵ; moreover,

$$c_\epsilon\geq\inf_{u\in\partial B_{\rho_0}(0)\cap H^+} I_{\lambda,\mu}(u)\geq\alpha_0>0$$

(see Theorem 2.12 in [16]). By Lemma 2.1 and Lemma 2.3, we infer that there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function $u \in H_0^1(\Omega)$, such that

$$u_n \longrightarrow u \quad \text{strongly in } H_0^1(\Omega),$$

and then c_ϵ is a critical value of $I_{\lambda,\mu}$ and u is a nontrivial solution of problem (1.1). ∎

3. Proof of Theorem 1.2

In this section, we consider Case II: $Q(0) < Q_M(\frac{S_\mu}{S_0})^{\frac{N}{N-2}}$. Observe that $S_\mu \leq S_0$; we easily infer that $a_i \neq 0$ $(1 \leq i \leq k)$, where $a_i \in \Omega$ satisfies $Q(a_i) = Q_M = \max_{\overline{\Omega}} Q(x)$. So $B_{\frac{2}{m}}(a_i) \subset \Omega$ for m large enough. Set

$$H_0^- = span\{e_{0,1}, e_{0,2}, \dots, e_{0,k}\}, H_0^+ = (H_0^-)^\perp,$$

where $e_{0,i}$ $(i = 1, 2, \dots)$ are the eigenfunctions $e_{\mu,i}$ for $\mu = 0$ in section 1.

Fix k; define the space

$$H_{0,m}^- = span\{e_{0,1}^m, e_{0,2}^m, \dots, e_{0,k}^m\},$$

where $e_{0,j}^m = \zeta_m e_{0,j}$ $(j = 1, 2, \dots)$,

$$\zeta_m(x) = \begin{cases} 0 & \text{if } x \in B_{\frac{1}{m}}(a_i), \\ m|x - a_i| - 1 & \text{if } x \in B_{\frac{2}{m}}(a_i) \backslash B_{\frac{1}{m}}(a_i), \\ 1 & \text{if } x \in \Omega \backslash B_{\frac{2}{m}}(a_i). \end{cases}$$

For any $m > 0, \epsilon > 0$, we define

$$v_{\epsilon,a_i}^m(x) = \begin{cases} U_{0,\epsilon}(x - a_i) - \frac{(\epsilon^2 N(N-2))^{\frac{N-2}{4}}}{(\epsilon^2 + (\frac{1}{m})^2)\sqrt{\mu}} & \text{if } x \in B_{\frac{1}{m}}(a_i), \\ 0 & \text{if } x \in \Omega \backslash B_{\frac{1}{m}}(a_i). \end{cases}$$

The following estimates hold:

$$\int_\Omega |\nabla v_{\epsilon,a_i}^m|^2 dx \leq S_0^{\frac{N}{2}} + C\epsilon^{N-2} m^{N-2}, \tag{3.1}$$

$$\int_\Omega |v_{\epsilon,a_i}^m|^{2^*} dx \geq S_0^{\frac{N}{2}} - C\epsilon^N m^N. \tag{3.2}$$

In fact, choosing $\mu = 0$ in (2.9) and (2.10) respectively, we get (3.1) and (3.2) immediately.

Set

$$c_\epsilon^* = \inf_{h \in \Gamma_{\epsilon,m}^*} \max_{u \in Q_{\epsilon,m}^*} I_{\lambda,\mu}(h(u)),$$

where

$$\Gamma^*_{\epsilon,m} = \{h \in C(Q^*_{\epsilon,m}, H^1_0(\Omega)) |\, h(u) = u, \forall u \in \partial Q^*_{\epsilon,m}\}$$

and

$$Q^*_{\epsilon,m} = (\overline{B_R(a_i)} \cap H^-_{0,m}) \oplus \{rv^m_{\epsilon,a_i} |\, 0 \le r \le R\}.$$

Then we have the following:

LEMMA 3.1: *Assume that $N \geq 5$, $\mu \geq 0$ and the assumption of (H_2) holds. Then for any $\lambda > 0$, $c^*_\epsilon < S_0^{\frac{N}{2}} / N Q_M^{\frac{N-2}{2}}$.*

Proof: As in the proof of Lemma 2.3, we suppose $\lambda_{0,k} \leq \lambda < \lambda_{0,k+1}$ for some integer k. Let $\max_{u \in Q^*_{\epsilon,m}} I_{\lambda,\mu}(u) = I_{\lambda,\mu}(w^m_0 + t^m_{0,\epsilon} v^m_{\epsilon,a_i})$, where $w^m_0 \in H^-_{0,m}$. By (ii) of Lemma 2.2 (the case: $\mu = 0$), we derive

(3.3)

$$\begin{aligned}
I_{\lambda,\mu}(w^m_0) =& \frac{1}{2}\int_\Omega \Big(|\nabla w^m_0|^2 - \mu \frac{(w^m_0)^2}{|x|^2} - \lambda (w^m_0)^2 \Big) dx - \frac{1}{2^*}\int_\Omega Q(x) |w^m_0|^{2^*} dx \\
\leq& \frac{\lambda_{0,k} - \lambda}{2} \int_\Omega (w^m_0)^2 dx + C m^{-(N-2)} \int_\Omega (w^m_0)^2 dx - \frac{1}{2^*} \min_{\overline{\Omega}} Q(x) \\
& \int_\Omega |w^m_0|^{2^*} dx \\
\leq& C m^{-(N-2)} \| w^m_0 \|^2_{L^{2^*}(\Omega)} - \frac{1}{2^*} \min_{\overline{\Omega}} Q(x) \| w^m_0 \|^{2^*}_{L^{2^*}(\Omega)} \\
\leq& \max_{t \geq 0} (C m^{-(N-2)} t^2 - \frac{1}{2^*} \min_{\overline{\Omega}} Q(x) t^{2^*}) \\
\leq& C m^{-\frac{N(N-2)}{2}}.
\end{aligned}$$

On the other hand, choosing $\mu = 0$ in (2.12), (2.13), and $\epsilon = m^{-(N+2)/2}$, we get as $m \longrightarrow \infty$

$$\int_\Omega |\nabla v^m_{\epsilon,a_i}|^2 dx \leq S_0^{\frac{N}{2}} + C m^{-\frac{N(N-2)}{2}}, \tag{3.4}$$

$$\int_\Omega |v^m_{\epsilon,a_i}|^{2^*} dx \geq S_0^{\frac{N}{2}} - C m^{-\frac{N^2}{2}}. \tag{3.5}$$

From the assumption of (H_2), and after a direct calculation, we get

$$\int_\Omega Q(x) |v^m_{\epsilon,a_i}|^{2^*} dx \geq Q(a_i) S_0^{\frac{N}{2}} - C m^{-\frac{N^2}{2}}. \tag{3.6}$$

In addition,

$$\int_\Omega |v^m_{\epsilon,a_i}|^2 dx \geq C m^{-(N+2)}. \tag{3.7}$$

Observe that $id \in \Gamma^*_{\epsilon,m}$ and $|suppw_0^m \cap suppv^m_{\epsilon,a_i}| = 0$. We deduce from (3.3)–(3.7) that

$$\begin{aligned}
c^*_\epsilon \le & \max_{u\in Q^*_{\epsilon,m}} I_{\lambda,\mu}(u)\\
=& I_{\lambda,\mu}(w_0^m + t^m_{0,\epsilon} v^m_{\epsilon,a_i})\\
=& I_{\lambda,\mu}(w_0^m) + I_{\lambda,\mu}(t^m_{0,\epsilon} v^m_{\epsilon,a_i})\\
\le & Cm^{-\frac{N(N-2)}{2}} + \frac{(t^m_{0,\epsilon})^2}{2}\int_\Omega (|\nabla v^m_{\epsilon,a_i}|^2 - \lambda (v^m_{\epsilon,a_i})^2)dx\\
& - \frac{(t^m_{0,\epsilon})^{2^*}}{2^*}\int_\Omega Q(x)|v^m_{\epsilon,a_i}|^{2^*}dx\\
\le & Cm^{-\frac{N(N-2)}{2}} + \frac{(t^m_{0,\epsilon})^2}{2}(S_0^{\frac{N}{2}} + Cm^{-\frac{N(N-2)}{2}} - \lambda C m^{-(N+2)})\\
& - \frac{(t^m_{0,\epsilon})^{2^*}}{2^*}(Q(a_i)S_0^{\frac{N}{2}} - Cm^{-\frac{N^2}{2}})\\
\le & Cm^{-\frac{N(N-2)}{2}} + \frac{1}{N}(S_0^{\frac{N}{2}} + Cm^{-\frac{N(N-2)}{2}} - \lambda C m^{-(N+2)})\\
& \times \left(\frac{S_0^{\frac{N}{2}} + Cm^{-\frac{N(N-2)}{2}} - \lambda Cm^{-(N+2)}}{Q(a_i)S_0^{\frac{N}{2}} - Cm^{-\frac{N^2}{2}}}\right)^{\frac{N-2}{2}},
\end{aligned}$$

Note that for $N \ge 5$, $N+2 < N(N-2)/2 < N^2/2$. Hence, for m large enough, we derive that

$$c^*_\epsilon \le \frac{S_0^{\frac{N}{2}}}{NQ(a_i)^{\frac{N-2}{2}}} + Cm^{-\frac{N(N-2)}{2}} - Cm^{-(N+2)} < \frac{S_0^{\frac{N}{2}}}{NQ(a_i)^{\frac{N-2}{2}}}. \qquad \blacksquare$$

Proof of Theorem 1.2: From [9], for m, R large enough $I_{\lambda,\mu}$ satisfies all the assumptions of the linking theorem [13]. Namely,

(i) There exist $\alpha, \rho > 0$ such that

$$I_{\lambda,\mu}(v) \ge \alpha \quad \forall v \in \partial B_\rho(a_i) \cap H_0^+.$$

(ii) There exists $R > \rho$ such that

$$I_{\lambda,\mu}|_{\partial Q^*_{\epsilon,m}} \le p(m) \quad \text{with } p(m) \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Moreover, $\partial B_\rho(a_i) \cap H_0^+$ and $\partial Q^*_{\epsilon,m}$ link (cf. [13]). Then we obtain a Palais–Smale sequence $\{v_n\}$ for $I_{\lambda,\mu}$ at level c^*_ϵ; moreover,

$$c^*_\epsilon \ge \inf_{v\in\partial B_\rho(a_i)\cap H_0^+} I_{\lambda,\mu}(v) \ge \alpha > 0$$

(see Theorem 2.12 in [16]). By Lemma 2.1 and Lemma 3.1, up to a subsequence, we may assume that

$$v_n \longrightarrow v \text{ strongly in } H_0^1(\Omega),$$

and then c_ϵ^* is a critical value of $I_{\lambda,\mu}$ and v is a solution of problem (1.1). ∎

4. Proof of Theorem 1.3

In this section, we first give some preliminary notation and useful lemmas.

Choosing $r_0 > 0$ small enough such that $0 \notin B_{r_0}(a_i)$, $B_{r_0}(a_i) \subset \Omega$ and $B_{r_0}(a_i) \cap B_{r_0}(a_j) = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, k$.

Define

$$g_i(u) = \frac{\int_\Omega \psi_i(x)|\nabla u|^2 dx}{\int_\Omega |\nabla u|^2 dx}, \quad \psi_i(x) = \min\{1, |x - a_i|\}, \quad 1 \le i \le k.$$

Then we have the following separation result.

LEMMA 4.1: *If $g_i(u) \le r_0/3$ and $g_j(u) \le r_0/3$ for $u \in H_0^1(\Omega)\backslash\{0\}$, then $i = j$.*

Proof: For any $u \in H_0^1(\Omega)\backslash\{0\}$ satisfying $g_i(u) \le r_0/3$ $(1 \le i \le k)$, we have

$$\frac{r_0}{3}\int_\Omega |\nabla u|^2 dx \ge \int_\Omega \psi_i(x)|\nabla u|^2 dx \ge \int_{\Omega\backslash B_{r_0}(a_i)} \psi_i(x)|\nabla u|^2 dx$$
$$\ge r_0 \int_{\Omega\backslash B_{r_0}(a_i)} |\nabla u|^2 dx,$$

which implies that

$$\int_\Omega |\nabla u|^2 dx \ge 3\int_{\Omega\backslash B_{r_0}(a_i)} |\nabla u|^2 dx, \quad 1 \le i \le k. \tag{4.1}$$

Hence, from (4.1), we obtain

$$2\int_\Omega |\nabla u|^2 dx \ge 3\Big(\int_{\Omega\backslash B_{r_0}(a_i)} |\nabla u|^2 dx + \int_{\Omega\backslash B_{r_0}(a_j)} |\nabla u|^2 dx\Big)$$
$$\ge 3\int_\Omega |\nabla u|^2 dx \quad \text{if } i \neq j,$$

which is a contradiction. ∎

Set

$$\mathcal{N}(\lambda, \mu) = \{u \in H_0^1(\Omega)\backslash\{0\} |\ \langle dI_{\lambda,\mu}(u), u\rangle = 0\},$$
$$\mathcal{N}_i(\lambda, \mu) = \{u \in \mathcal{N}(\lambda, \mu) |\ g_i(u) < r_0/3\},$$

and

$$\mathcal{O}_i(\lambda,\mu)=\{u\in\mathcal{N}(\lambda,\mu)|\ g_i(u)=r_0/3\}.$$

Define

$$c_i(\lambda,\mu):=\inf_{u\in\mathcal{N}_i(\lambda,\mu)} I_{\lambda,\mu}(u)\quad\text{and}\quad \overline{c_i}(\lambda,\mu):=\inf_{u\in\mathcal{O}_i(\lambda,\mu)} I_{\lambda,\mu}(u),\qquad i=1,2,\dots,k.$$

Then we have

LEMMA 4.2: $c_i(\lambda,\mu)<S_0^{\frac{N}{2}}/NQ_M^{\frac{N-2}{2}}$.

Proof: Let $\rho>0$ be small enough such that $0\notin B_\rho(a_i)$ for $i=1,2,\dots,k$, and $B_\rho(a_i)\subset\Omega$. Set $w_\epsilon^{a_i}(x)=\varphi(x)W_\epsilon^{a_i}(x)$, where

$$W_\epsilon^{a_i}(x)=\frac{(N(N-2)\epsilon)^{\frac{N-2}{4}}}{(\epsilon+|x-a_i|^2)^{\frac{N-2}{2}}}\quad\text{and}\quad 0\le\varphi\le 1,\quad \varphi(x)=\begin{cases}1 & \text{if } |x-a_i|\le\frac{\rho}{2},\\ 0 & \text{if } |x-a_i|\ge\rho.\end{cases}$$

Then we have $t_\epsilon^{a_i}w_\epsilon^{a_i}\in\mathcal{N}(\lambda,\mu)$, where

$$t_\epsilon^{a_i}=\Big(\frac{\int_\Omega(|\nabla w_\epsilon^{a_i}|^2-\mu\frac{|w_\epsilon^{a_i}|^2}{|x|^2}-\lambda|w_\epsilon^{a_i}|^2)dx}{\int_\Omega Q(x)|w_\epsilon^{a_i}|^{2^*}dx}\Big)^{\frac{N-2}{4}}.$$

Furthermore,

$$\begin{aligned}g_i(t_\epsilon^{a_i}w_\epsilon^{a_i})&=\frac{\int_\Omega\psi_i(x)|\nabla w_\epsilon^{a_i}(x)|^2dx}{\int_\Omega|\nabla w_\epsilon^{a_i}(x)|^2dx}\\&=\frac{\int_{\frac{\Omega-a_i}{\epsilon}}\psi_i(a_i+\epsilon y)|\nabla(\varphi(a_i+\epsilon y)W_1^0(y))|^2dy}{\int_{\frac{\Omega-a_i}{\epsilon}}|\nabla(\varphi(a_i+\epsilon y)W_1^0(y))|^2dy}\\&\longrightarrow\psi_i(a_i)=0\quad\text{as }\epsilon\longrightarrow 0.\end{aligned}$$

Hence, there exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, $g_i(t_\epsilon^{a_i}w_\epsilon^{a_i})<r_0/3$, which implies $t_\epsilon^{a_i}w_\epsilon^{a_i}\in\mathcal{N}_i(\lambda,\mu)$, $1\le i\le k$. Therefore, we get

$$\begin{aligned}c_i(\lambda,\mu)&\le I_{\lambda,\mu}(t_\epsilon^{a_i}w_\epsilon^{a_i})=\max_{t\ge0}I_{\lambda,\mu}(tw_\epsilon^{a_i})\\&=\Big(\frac{\int_\Omega(|\nabla w_\epsilon^{a_i}|^2-\mu\frac{|w_\epsilon^{a_i}|^2}{|x|^2}-\lambda|w_\epsilon^{a_i}|^2)dx}{(\int_\Omega Q(x)|w_\epsilon^{a_i}|^{2^*}dx)^{\frac{2}{2^*}}}\Big)^{\frac{N}{2}}.\end{aligned}\tag{4.2}$$

From [2], we know that the following estimates hold:

$$\int_\Omega|\nabla w_\epsilon^{a_i}|^2dx=\int_{R^N}|\nabla W_1^0|^2dx+O(\epsilon^{\frac{N-2}{2}}),\tag{4.3}$$

$$\int_\Omega|w_\epsilon^{a_i}|^{2^*}dx=\int_{R^N}|W_1^0|^{2^*}dx+O(\epsilon^{\frac{N}{2}}),\tag{4.4}$$

$$\int_\Omega|w_\epsilon^{a_i}|^2dx=L(\epsilon)=\begin{cases}C\epsilon+O(\epsilon^{\frac{N-2}{2}}) & \text{if } N\ge5,\\ C\epsilon|\log\epsilon|+O(\epsilon) & \text{if } N=4,\end{cases}\tag{4.5}$$

To proceed further, we need to estimate the two terms in (4.2):

$$\int_\Omega \frac{|w_\epsilon^{a_i}|^2}{|x|^2}dx \quad \text{and} \quad \int_\Omega Q(x)|w_\epsilon^{a_i}|^{2^*}dx.$$

$$\begin{aligned}
\int_\Omega \frac{|w_\epsilon^{a_i}|^2}{|x|^2}dx &\geq C\epsilon^{\frac{N-2}{2}} \int_{B_{\frac{\rho}{2}}(a_i)} \frac{dx}{|x|^2(\epsilon+|x-a_i|^2)^{N-2}} \\
&\geq C\epsilon^{\frac{N-2}{2}} \int_{B_{\frac{\rho}{2}}(0)} \frac{dy}{|y+a_i|^2(\epsilon+|y|^2)^{N-2}} \\
&\geq C\epsilon^{\frac{N-2}{2}} \int_{B_{\frac{\rho}{2}}(0)} \frac{dy}{(|y|^2+|a_i|^2)(\epsilon+|y|^2)^{N-2}} \qquad (4.6)\\
&\geq C\epsilon^{\frac{N-2}{2}} \int_0^{\frac{\rho}{2}} \frac{r^{N-1}}{(\epsilon+r^2)^{N-2}} \\
&\geq C\epsilon.
\end{aligned}$$

It follows from the assumption of (H_2) that for any $\eta > 0$, there exists $\rho > 0$ small enough such that for $x \in B_\rho(a_i)$, $|Q(x) - Q(a_i)| \leq \eta|x-a_i|^2$. So we have

$$\begin{aligned}
\left| \int_\Omega (Q(x)-Q(a_i))|w_\epsilon^{a_i}|^{2^*}dx \right| &\leq \int_{B_\rho(a_i)} |Q(x)-Q(a_i)||w_\epsilon^{a_i}|^{2^*}dx \\
&\leq C\eta\epsilon^{\frac{N}{2}} \int_{B_\rho(a_i)} \frac{|x-a_i|^2}{(\epsilon+|x-a_i|^2)^N}dx \\
&\leq C\eta\epsilon^{\frac{N}{2}} \int_0^\rho \frac{r^{N+1}}{(\epsilon+r^2)^N}dr \\
&\leq C\eta\epsilon \int_0^{\frac{\rho}{\sqrt{\epsilon}}} \frac{t^{N+1}}{(1+t^2)^N}dt \\
&\leq C\eta\epsilon,
\end{aligned}$$

which implies

$$\int_\Omega (Q(x)-Q(a_i))|w_\epsilon^{a_i}|^{2^*}dx = o(\epsilon). \qquad (4.7)$$

Thus, from (4.7), we derive

$$\begin{aligned}\int_\Omega Q(x)|w_\epsilon^{a_i}|^{2^*}dx =&Q_M\int_{R^N}|W_\epsilon^{a_i}|^{2^*}dx - Q_M\int_{R^N\setminus\Omega}|W_\epsilon^{a_i}|^{2^*}dx\\ &+Q_M\int_\Omega(|\varphi|^{2^*}-1)|W_\epsilon^{a_i}|^{2^*}dx\\ &+\int_\Omega(Q(x)-Q(a_i))|w_\epsilon^{a_i}|^{2^*}dx\\ =&Q_M\int_{R^N}|W_1^0|^{2^*}dx+O(\epsilon^{\frac{N}{2}})+o(\epsilon)\\ =&Q_M\int_{R^N}|W_1^0|^{2^*}dx+o(\epsilon).\end{aligned}\tag{4.8}$$

Inserting (4.3), (4.5), (4.6) and (4.8) into (4.2), we deduce that for $\epsilon > 0$ small enough

$$\begin{aligned}c_i(\lambda,\mu)&\leq\frac{1}{N}\Big(\frac{\int_{R^N}|\nabla W_1^0|^2dx+O(\epsilon^{\frac{N-2}{2}})-C\epsilon-L(\epsilon)}{(Q_M\int_{R^N}|W_1^0|^{2^*}dx+o(\epsilon))^{\frac{2}{2^*}}}\Big)^{\frac{N}{2}}\\ &\leq\frac{S_0^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}(1+O(\epsilon^{\frac{N-2}{2}})-C\epsilon-CL(\epsilon))^{\frac{N}{2}}\\ &<\frac{S_0^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}.\end{aligned}$$ ∎

LEMMA 4.3: *There exist $\lambda_0,\mu_0>0$ such that*

$$\overline{c_i}(\lambda,\mu)>\frac{S_0^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}\quad \textit{for all } \lambda\in(0,\lambda_0) \textit{ and } \mu\in(0,\mu_0).$$

Proof: Suppose to the contrary that we could find two positive sequences $\lambda_n \longrightarrow 0$ and $\mu_n \longrightarrow 0$ as $n\longrightarrow\infty$, such that $\overline{c_i}(\lambda_n,\mu_n)\longrightarrow c\leq S_0^{\frac{N}{2}}/NQ_M^{\frac{N-2}{2}}$. Consequently, there exists $u_n\in\mathcal{O}_i(\lambda_n,\mu_n)$ such that as $n\longrightarrow\infty$,

$$I_{\lambda_n,\mu_n}(u_n)\longrightarrow c$$

and

$$\int_\Omega(|\nabla u_n|^2-\mu_n\frac{|u_n|^2}{|x|^2}-\lambda_n|u_n|^2)dx=\int_\Omega Q(x)|u_n|^{2^*}dx.\tag{4.9}$$

It then follows easily that $|\,u_n|\;\leq C$, and in particular,

$$\lim_{n\longrightarrow\infty}\mu_n\int_\Omega\frac{|u_n|^2}{|x|^2}dx\leq\lim_{n\longrightarrow\infty}\frac{\mu_n}{\bar\mu}\int_\Omega|\nabla u_n|^2dx=0\quad\text{and}$$

$$\lim_{n\longrightarrow\infty}\lambda_n\int_\Omega|u_n|^2dx=0.$$

From (4.9), and by the Hölder and Sobolev inequalities, we can fix $m_0 > 0$ such that

$$\int_\Omega |\nabla u_n|^2 dx \geq m_0 \quad \text{and} \quad \int_\Omega Q(x)|u_n|^{2^*} dx \geq m_0.$$

Thus, up to a subsequence, we infer that

$$\lim_{n\longrightarrow\infty} \int_\Omega |\nabla u_n|^2 dx = \lim_{n\longrightarrow\infty} \int_\Omega Q(x)|u_n|^{2^*} dx = a > 0.$$

Furthermore, we deduce

$$\begin{aligned} a &\leq Q_M \lim_{n\longrightarrow\infty} \int_\Omega |u_n|^{2^*} dx \leq Q_M S_0^{-\frac{2^*}{2}} \lim_{n\longrightarrow\infty} \left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{2^*}{2}} \\ &\leq Q_M S_0^{-\frac{2^*}{2}} a^{\frac{2^*}{2}}. \end{aligned} \tag{4.10}$$

Thus we get

$$a \geq S_0^{\frac{N}{2}} / Q_M^{\frac{N-2}{2}}. \tag{4.11}$$

On the other hand, we have as $n \longrightarrow \infty$

(4.12)

$$\begin{aligned} \frac{1}{N} a &= \frac{1}{2}\int_\Omega (|\nabla u_n|^2 - \mu_n \frac{|u_n|^2}{|x|^2} - \lambda_n |u_n|^2) dx - \frac{1}{2^*}\int_\Omega Q(x)|u_n|^{2^*} dx + o(1) \\ &= I_{\lambda_n,\mu_n}(u_n) + o(1) \\ &\leq \frac{S_0^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}}. \end{aligned}$$

Hence, from (4.11) and (4.12), we infer $a = S_0^{\frac{N}{2}} / Q_M^{\frac{N-2}{2}}$, and then from (4.10)

$$\lim_{n\longrightarrow\infty} \int_\Omega Q_M |u_n|^{2^*} dx = S_0^{\frac{N}{2}} / Q_M^{\frac{N-2}{2}}.$$

Therefore,

$$\lim_{n\longrightarrow\infty} \int_\Omega (Q_M - Q(x))|u_n|^{2^*} dx = 0. \tag{4.13}$$

Set $w_n = u_n / |u_n|_{L^{2^*}(\Omega)}$; then $|w_n|_{L^{2^*}(\Omega)} = 1$, and

$$\lim_{n\longrightarrow\infty} \int_\Omega |\nabla w_n|^2 dx = \lim_{n\longrightarrow\infty} \frac{\int_\Omega |\nabla u_n|^2 dx}{|u_n|^2_{L^{2^*}(\Omega)}} = S_0.$$

That is, $\{w_n\}$ is a minimizing sequence for the problem

$$S_0 := \inf\left\{ \int_\Omega |\nabla u|^2 dx \mid u \in H_0^1(\Omega)\backslash\{0\}, \int_\Omega |u|^{2^*} dx = 1 \right\}.$$

We now use a result of P. L. Lions [12] to conclude that there exists an $x_0 \in \overline{\Omega}$ and a subsequence, still denoted by $\{w_n\}$, such that

$$|\nabla w_n|^2 \rightharpoonup d\widetilde{\mu} = S_0 \delta_{x_0} \text{ weakly in the sense of measure,}$$

and

$$|w_n|^{2^*} \rightharpoonup d\widetilde{\nu} = \delta_{x_0} \text{ weakly in the sense of measure.}$$

Observe that $g_i(w_n) = g_i(u_n) = r_0/3$; we conclude that

$$\frac{r_0}{3} = \lim_{n \longrightarrow \infty} g_i(w_n) = \lim_{n \longrightarrow \infty} \frac{\int_\Omega \psi_i(x) |\nabla w_n|^2 dx}{\int_\Omega |\nabla w_n|^2 dx} = \psi_i(x_0),$$

which implies that $x_0 \notin \{a_i | \, i = 1, 2, \ldots, k\}$. Therefore, from (4.13), we deduce

$$Q_M = \lim_{n \longrightarrow \infty} \int_\Omega Q_M |w_n|^{2^*} dx = \lim_{n \longrightarrow \infty} \int_\Omega Q(x) |w_n|^{2^*} dx = Q(x_0),$$

which is impossible, because that Q is not a constant function. ∎

LEMMA 4.4: *For any $u \in \mathcal{N}_i(\lambda, \mu)(1 \leq i \leq k)$, there exists $\rho_u > 0$ and a differentiable function f: $B_{\rho_u}(0) \subset H_0^1(\Omega) \longrightarrow \mathbb{R}$ such that $f(0) = 1$, and for any $w \in B_{\rho_u}(0)$, we have $f(w)(u - w) \in \mathcal{N}_i(\lambda, \mu)$. Moreover, for all $v \in H_0^1(\Omega)$,*

$$\langle f'(0), v \rangle = \frac{2 \int_\Omega (\nabla u \cdot \nabla v - \mu \frac{uv}{|x|^2} - \lambda uv) dx - 2^* \int_\Omega Q(x) |u|^{2^* - 2} uv dx}{\int_\Omega (|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} - \lambda |u|^2) dx - (2^* - 1) \int_\Omega Q(x) |u|^{2^*} dx}.$$

Proof: Let $u \in \mathcal{N}_i(\lambda, \mu)$ and G: $\mathbb{R}^+ \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ be the function defined by

$$G(t, w) = t \int_\Omega (|\nabla(u - w)|^2 - \mu \frac{|u - w|^2}{|x|^2} - \lambda |u - w|^2) dx - t^{2^* - 1} \int_\Omega Q(x) |u - w|^{2^*} dx.$$

Then $G(1, 0) = 0$ and

$$\begin{aligned} G_t(1, 0) &= \int_\Omega \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} - \lambda |u|^2 \right) dx - (2^* - 1) \int_\Omega Q(x) |u|^{2^*} dx \\ &= (2 - 2^*) \int_\Omega Q(x) |u|^{2^*} dx \\ &\neq 0. \end{aligned}$$

Hence, by the implicit function theorem, we infer that there exists $\rho_u > 0$ small enough and a differentiable function $f\colon B_{\rho_u}(0) \subset H_0^1(\Omega) \longrightarrow \mathbb{R}$ such that $f(0) = 1$ and $G(f(w), w) = 0$ for all $w \in B_{\rho_u}(0)$. It is easy to verify from $G(f(w), w) = 0$ that $f(w)(u - w) \in \mathcal{N}_i(\lambda, \mu)$ and

$$\begin{aligned}\langle f'(0), v\rangle &= -\frac{\langle G_w(1,0), v\rangle}{G_t(1,0)}\\ &= \frac{2\int_\Omega(\nabla u\cdot\nabla v - \mu\frac{uv}{|x|^2} - \lambda uv)dx - 2^*\int_\Omega Q(x)|u|^{2^*-2}uvdx}{\int_\Omega(|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2} - \lambda|u|^2)dx - (2^*-1)\int_\Omega Q(x)|u|^{2^*}dx}.\end{aligned}$$ ∎

Proof of Theorem 1.3: From Lemmas 4.2 and 4.3, we conclude that

$$c_i(\lambda,\mu) < \overline{c_i}(\lambda,\mu)\ (1 \le i \le k) \quad \text{for all } \lambda \in (0,\lambda_0) \text{ and } \mu \in (0,\mu_0). \tag{4.14}$$

It then follows that

$$c_i(\lambda,\mu) = \inf\{I_{\lambda,\mu}(u) |\ u \in (\mathcal{N}_i(\lambda,\mu) \cup \mathcal{O}_i(\lambda,\mu))\}.$$

Let $\{u_n^i\} \subset (\mathcal{N}_i(\lambda,\mu) \cup \mathcal{O}_i(\lambda,\mu))$ be a minimizing sequence for $c_i(\lambda,\mu)$. By replacing u_n^i with $|u_n^i|$, if necessary, we may assume that $u_n^i \ge 0$. By Ekeland's variational principle [7], there exists a subsequence, still denoted by $\{u_n^i\}$, such that

$$I_{\lambda,\mu}(u_n^i) \le c_i(\lambda,\mu) + \frac{1}{n},$$

and

$$I_{\lambda,\mu}(w) \ge I_{\lambda,\mu}(u_n^i) - \frac{1}{n}\|w - u_n^i\| \quad \text{for all } w \in (\mathcal{N}_i(\lambda,\mu) \cup \mathcal{O}_i(\lambda,\mu)).$$

From (4.14), we may assume that $u_n^i \in \mathcal{N}_i(\lambda,\mu)$ for sufficiently large n. Set $v_\rho = \rho v$ with $\|v\| = 1$ and $0 < \rho < \rho_{u_n^i}$; then $v_\rho \in B_{\rho_{u_n^i}}(0)$, and from Lemma 4.4, $w_\rho = f_{u_n^i}(v_\rho)(u_n^i - v_\rho) \in \mathcal{N}_i(\lambda,\mu)$, where $\rho_{u_n^i}, f_{u_n^i}$ are from Lemma 4.4. Observe that $f_{u_n^i}(v_\rho) \longrightarrow f_{u_n^i}(1) = 1$ as $\rho \longrightarrow 0$, and by a Taylor expansion, we obtain

$$\begin{aligned}\frac{1}{n}\|w_\rho - u_n^i\| &\ge I_{\lambda,\mu}(u_n^i) - I_{\lambda,\mu}(w_\rho)\\ &= \langle dI_{\lambda,\mu}(u_n^i), u_n^i - w_\rho\rangle + o(\|u_n^i - w_\rho\|)\\ &= \rho f_{u_n^i}(\rho v)\langle dI_{\lambda,\mu}(u_n^i), v\rangle + (1 - f_{u_n^i}(\rho v))\langle dI_{\lambda,\mu}(u_n^i), u_n^i\rangle\\ &\quad + o(\|u_n^i - w_\rho\|)\\ &= \rho f_{u_n^i}(\rho v)\langle dI_{\lambda,\mu}(u_n^i), v\rangle + o(\|u_n^i - w_\rho\|).\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
|\langle dI_{\lambda,\mu}(u_n^i), v\rangle| &\le \frac{|\, w_\rho - u_n^i|\,(\frac{1}{n} + |o(1)|)}{\rho|f_{u_n^i}(\rho v)|} \\
&\le \frac{|\, u_n^i(f_{u_n^i}(\rho v) - f_{u_n^i}(0)) - \rho v f_{u_n^i}(\rho v)|\,(\frac{1}{n} + |o(1)|)}{\rho|f_{u_n^i}(\rho v)|} \\
&\le \frac{|\, u_n^i|\,|\, f_{u_n^i}(\rho v) - f_{u_n^i}(0)|\, + \rho|\, v|\, |f_{u_n^i}(\rho v)|}{\rho|f_{u_n^i}(\rho v)|}\Big(\frac{1}{n} + |o(1)|\Big) \\
&\le C(1 + |\, f'_{u_n^i}(0)|\,)\Big(\frac{1}{n} + |o(1)|\Big).
\end{aligned}$$

Therefore, we deduce that $dI_{\lambda,\mu}(u_n^i) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $\{u_n^i\}$ is a Palais–Smale sequence for $I_{\lambda,\mu}$ at the level $c_i(\lambda,\mu)$. Since $c_i(\lambda,\mu) < S_0^{\frac{N}{2}}/NQ_M^{\frac{N-2}{2}} = c^*$ in Case II, from Lemma 2.1, we infer that there is a subsequence of $\{u_n^i\}$, still denoted by $\{u_n^i\}$, and a function $u^i \in H_0^1(\Omega)$, such that

$$u_n^i \longrightarrow u^i \quad (1 \le i \le k) \quad \text{strongly in } H_0^1(\Omega),$$

and then $u^i \ge 0 (1 \le i \le k)$. By the strongly maximum principle, we obtain $u^i > 0$ $(1 \le i \le k)$ in Ω. Since $g_i(u^i) \in B_{\frac{r_0}{3}(a_i)}$, and $B_{\frac{r_0}{3}(a_i)}$ are disjoint for $i = 1, 2, \ldots, k$, we conclude from Lemma 4.1 that $u^i (1 \le i \le k)$ are distinct positive solutions of (1.1). ∎

ACKNOWLEDGEMENT: The author would like to thank Professor Daomin Cao for helpful discussions during the preparation for this paper.

References

[1] J. P. Garcia Azorero and I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, Journal of Differential Equations **144** (1998), 441–476.

[2] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent*, Communications on Pure and Applied Mathematics **36** (1983), 437–478.

[3] D. Cao and P. Han, *Solutions for semilinear elliptic equations with critical exponents and Hardy potential*, Journal of Differential Equations **205** (2004), 521–537.

[4] D. Cao and S. Peng, *A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms*, Journal of Differential Equations **193** (2003), 424–434.

[5] K. S. Chou and C. W. Chu, *On the best constant for a weighted Sobolev-Hardy inequality*, Journal of the London Mathematical Society **48** (1993), 137–151.

[6] E. Egnell, *Elliptic boundary value problems with singular coefficients and critical nonlinearities*, Indiana University Mathematics Journal **38** (1989), 235–251.

[7] I. Ekeland, *On the variational principle*, Journal of Mathematical Analysis and Applications **17** (1974), 324–353.

[8] I. Ekeland and N. Ghoussoub, *Selected new aspects of the calculus of variations in the large*, Bulletin of the American Mathematical Society **39** (2002), 207–265.

[9] A. Ferrero and F. Gazzola, *Existence of solutions for singular critical growth semilinear elliptic equations*, Journal of Differential Equations **177** (2001), 494–522.

[10] N. Ghoussoub and C. Yuan, *Multiple solutions for quasilinear PDEs involving critical Sobolev and Hardy exponents*, Transactions of the American Mathematical Society **352** (2000), 5703–5743.

[11] E. Jannelli, *The role played by space dimension in elliptic critical problems*, Journal of Differential Equations **156** (1999), 407–426.

[12] P. L. Lions, *The concentration-compactness principle in the calculus of variations: the limit case*, Revista Matemática Iberoamericana **1** (1985), 145–201; 45–121.

[13] P. Rabinowitz, *Minimax methods in critical points theory with applications to differential equations*, CBMS series, no. 65, Providence, RI, 1986.

[14] D. Ruiz and M. Willem, *Elliptic problems with critical exponents and Hardy potentials*, Journal of Differential Equations **190** (2003), 524–538.

[15] S. Terracini, *On positive solutions to a class equations with a singular coefficient and critical exponent*, Advances in Differential Equations **2** (1996), 241–264.

[16] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.